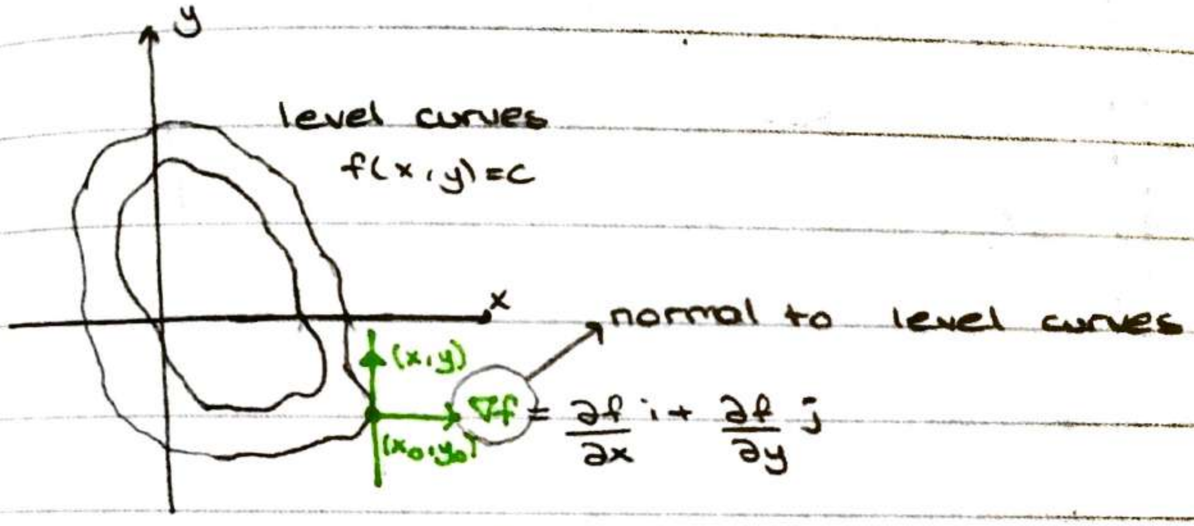
Last Time

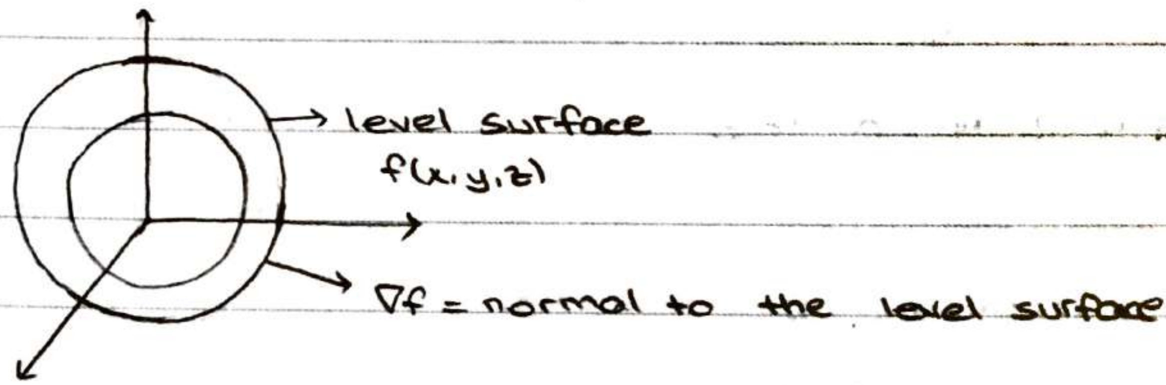
$z = f(x,y)$



$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$

$[(x-x_0)i + (y-y_0)j] \cdot \nabla f = 0$

$w = f(x,y,z)$

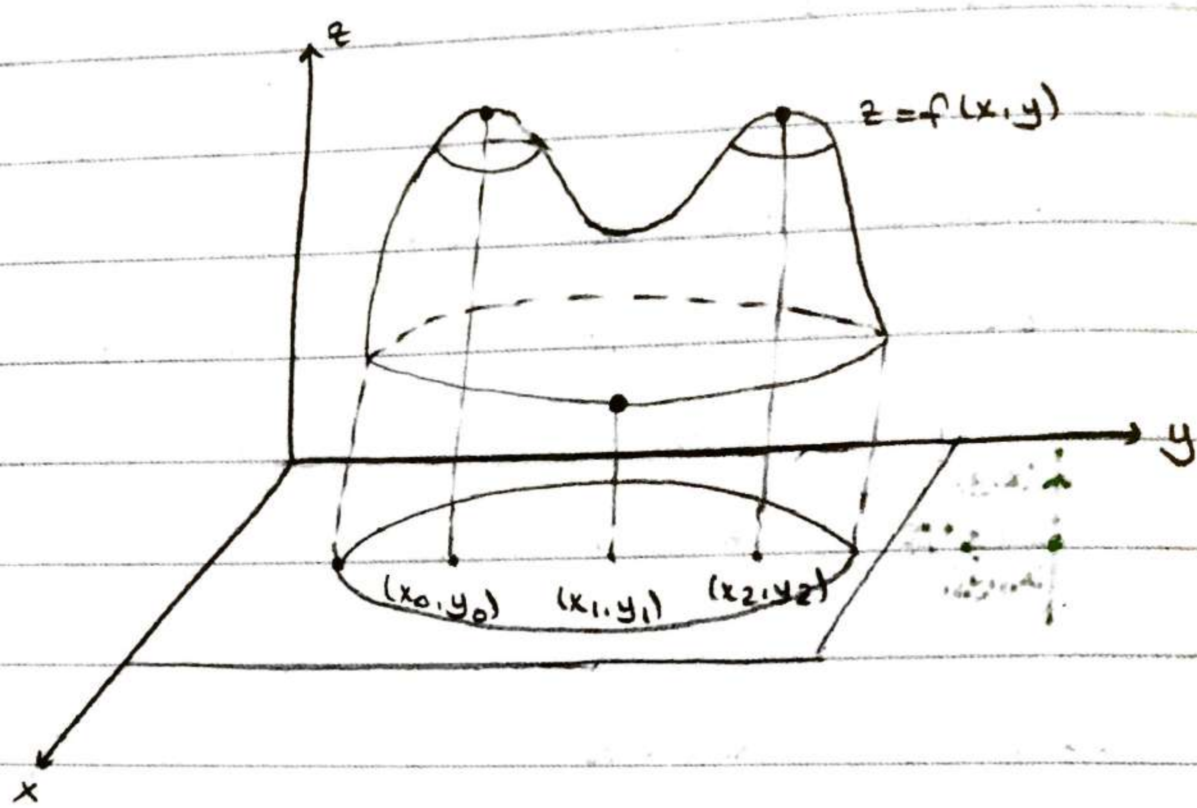


$\nabla f(x_0,y_0,z_0) = \frac{\partial f}{\partial x}(x_0,y_0,z_0)i + \frac{\partial f}{\partial y}(x_0,y_0,z_0)j + \frac{\partial f}{\partial z}(x_0,y_0,z_0)k$

- Tangent plane at (x_0,y_0,z_0)

$\frac{\partial f}{\partial x}(x_0,y_0,z_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0,z_0)(y-y_0) + \frac{\partial f}{\partial z}(x_0,y_0,z_0)(z-z_0) = 0$

14.7 → EXTREME VALUES AND SADDLE POINTS



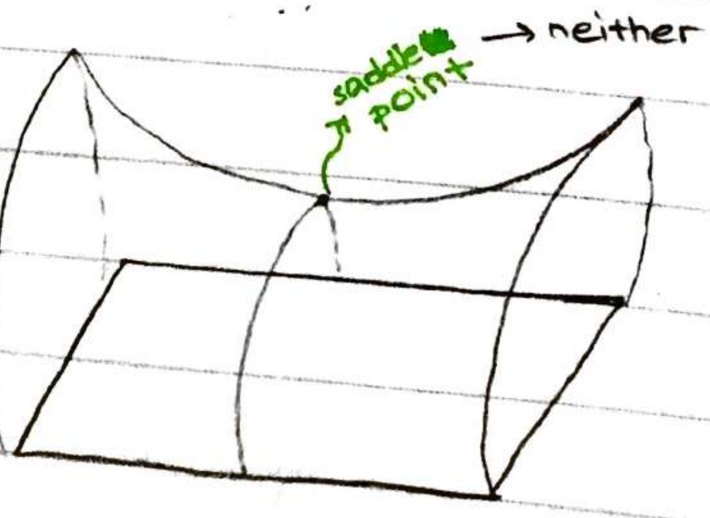
* f has local max at (x_0, y_0) and (x_2, y_2)
* f has local min at (x_1, y_1)

Theorem: If f has a local max or min at (x_0, y_0) and if partial derivatives of f exist at (x_0, y_0) then

$f_x(x_0, y_0) = 0$

$f_y(x_0, y_0) = 0$

If $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ or one of $f_x(x_0, y_0)$ $f_y(x_0, y_0)$ does not exist we call (x_0, y_0) a critical point of f



Theorem: Second derivative test

$$f_x(a,b) = f_y(a,b) = 0$$

a) f has a local max at (a,b) if $f_{xx} < 0$ and

$$\Delta = f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{at } (a,b)$$

b) f has a local min at (a,b) if $f_{xx} > 0$ and

$$\Delta > 0$$

c) f has a saddle point at (a,b) if

$$\Delta < 0$$

d) $\Delta = 0 \Rightarrow$ test is inconclusive

$$\Delta = f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

= called Hession or discriminant

Proofish:

$$f(x,y) = f(a,b) + \cancelto{0}{f_x(a,b)}(x-a) + \cancelto{0}{f_y(a,b)}(y-b)$$
$$+ \frac{1}{2}\Big(\underset{A}{f_{xx}(a,b)}\underset{x^2}{(x-a)^2} + 2\underset{B}{f_{xy}(a,b)}\underset{x}{(x-a)}\underset{y}{(y-b)} + \underset{C}{f_{yy}(a,b)}\underset{y^2}{(y-b)^2}\Big) + \ldots$$

$$f(x,y) - f(a,b) \approx \frac{1}{2!}(Ax^2 + 2Bxy + Cy^2)$$

$$= \frac{A}{2!}\left(x^2 + \frac{2B}{A}xy + \frac{C}{A}y^2\right)$$

$$= \frac{A}{2!}\left(\left(x - \frac{B}{A}y\right)^2 - \frac{B^2}{A^2}y^2 + \frac{C}{A}y^2\right)$$

$$= \frac{A}{2!}\left(\left(x - \frac{B}{A}y\right)^2 + \frac{AC - B^2}{A^2}y^2\right)$$

(book 3)

ex: $f(x,y) = x^2 + xy + 3x + 2y + 5$

Find all the local maxima, local minima, saddle points.

$$\left.\begin{aligned} f_x &= 2x + y + 3 = 0 \\ f_y &= x + 2 = 0 \end{aligned}\right\} \begin{aligned} &x = -2 \\ &-4 + y + 3 = 0 \rightarrow y = 1 \end{aligned}$$

Only one critical point : $(-2, 1)$

$f_{xx} = 2$

$f_{xy} = 1$ $\qquad \Delta = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = \underbrace{-1 < 0}$

$f_{yy} = 0$ $\qquad$ f has a saddle point at $(-2, 1)$

ex: $f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy$

same question.

$f_x = -6x + 6y = 0 \longrightarrow x = y$

$f_y = 6y - 6y^2 + 6x = 0$

$6y - 6y^2 + 6y = 0$

$2y - y^2 = 0$

$y(2 - y) = 0$

$y = 0 \rightarrow x = 0$

$y = 2 \rightarrow x = 2$

Two critical points $(0,0)$ and $(2,2)$

$f_{xx} = -6$

$f_{xy} = 6$

$f_{yy} = 6 - 12y$

$\Delta(0,0) = (-6) \cdot 6 - 6^2 < 0$

f has a saddle point at $(0,0)$

$\Delta(2,2) = (-6)(-18) - 6^2 > 0$

$f_{xx} < 0$ and $\Delta > 0 \Rightarrow f$ has a local max at $(2,2)$

ex: $f(x,y) = 10xye^{-(x^2+y^2)}$

same question.

$f_x = 10ye^{-(x^2+y^2)} + 10xy \cdot e^{-(x^2+y^2)} \cdot -2x$

$= 10ye^{-(x^2+y^2)}[1-2x^2] = 0$

$f_y = 10xe^{-(x^2+y^2)}[1-2y^2] = 0$

$y = 0 \qquad x = \mp \frac{1}{\sqrt{2}}$

$y = 0 \Rightarrow x = 0$

$x = \mp\frac{1}{\sqrt{2}} \Rightarrow y = \mp \frac{1}{\sqrt{2}}$

⊛ 5 critical points

$(0,0)$ $(1/\sqrt{2}, 1/\sqrt{2})$ $(1/\sqrt{2}, -1/\sqrt{2})$ $(-1/\sqrt{2}, 1/\sqrt{2})$ $(-1/\sqrt{2}, -1/\sqrt{2})$

$f_{xx} = 10ye^{-(x^2+y^2)} \cdot 2x \cdot [1-2x^2] + 10y \cdot e^{-(x^2+y^2)} \cdot (-4x)$

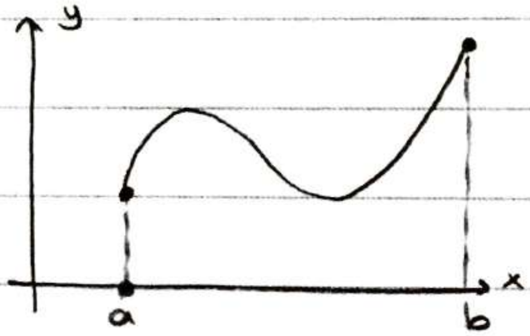
$= 10ye^{-(x^2+y^2)}[-2x+4x^3-4x] \Rightarrow f_{xx} = 20xye^{-(x^2+y^2)}(2x^2-3)$

$f_{yy} = 20xye^{-(x^2+y^2)}(2y^2-3)$

$f_{xy} = 10(1-2x^2)(1-2y^2)e^{-(x^2+y^2)}$

	f_{xx}	f_{xy}	f_{yy}	$\Delta \rightarrow f_{xx}f_{yy}-f_{xy}^2$	
$(0,0)$	0	10	0	-100	saddle
$(1/\sqrt{2}, 1/\sqrt{2})$	$-20/e$	0	$-20/e$	$400/e^2$	local max
$(-1/\sqrt{2}, 1/\sqrt{2})$	$20/e$	0	$20/e$	$400/e^2$	local min
$(1/\sqrt{2}, -1/\sqrt{2})$	$20/e$	0	$20/e$	$400/e^2$	local min
$(-1/\sqrt{2}, -1/\sqrt{2})$	$-20/e$	0	$-20/e$	$400/e^2$	local max

(Figure 14.49)



✓ If f is continuous on closed bounded interval it must attain its absolute max values at one of the following points

* where $f'(x) = 0$
* where $f'(x)$ is undefined
* at the end points of the interval where it is defined

Same holds true for functions of two variables defined on closed and bounded regions.

ex: Find the absolute max and absolute min values of $f(x,y) = 2 + 2x + 4y + x^2 - y^2$ on the triangular region in the first quadrant bounded by the lines $x=0$ $y=0$ and $y = 9-x$

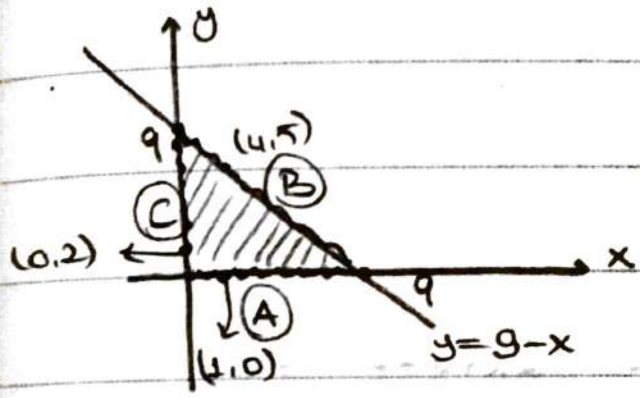
critical points

$f_x = 2 - 2x = 0 \quad \Rightarrow \quad x = 1$

$f_y = 4 - 2y = 0 \quad \Rightarrow \quad y = 2$

$(1,2) \rightarrow$ only critical point

boundary points



A: $y = 0 \quad , \quad 0 \le x \le 9$

$g(x) = f(x,0) = 2 + 2x - x^2 \quad , \quad 0 \le x \le 9$

$g'(x) = 2 - 2x = 0 \quad \Rightarrow \quad x = 1 \; , \; y = 0$

B: $y = 9 - x$

$g(x) = f(x, 9-x) = 2 + 2x + 4(9-x) - x^2 - (9-x)^2$

$g'(x) = 2 - 4 - 2x - 2(9-x)(-1)$

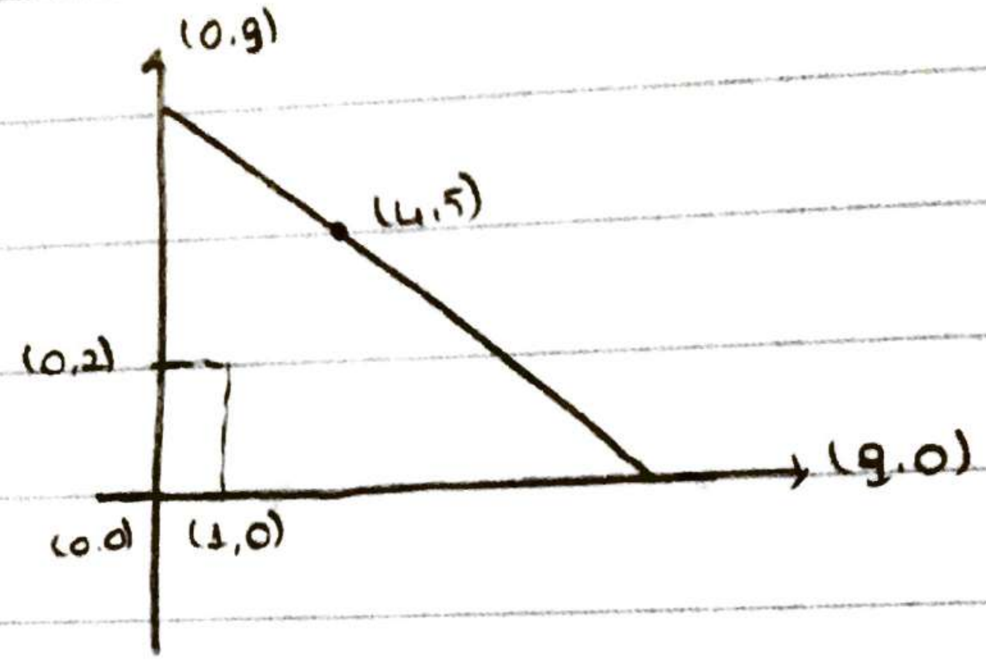
$= 2 - 2x + 18 - 2x$

$= 16 - 4x = 0 \quad \Rightarrow \quad x = 4 \quad y = 5$

C: $x = 0 \qquad 0 \le x \le 9$

$g(y) = f(0,y) = 2 + 4y - y^2$

$g'(y) = 4 - 2y = 0 \quad \Rightarrow \quad y = 2 \quad x = 0$



$f(0,0) = 2$

$f(1,0) = 3$

$f(9,0) = -61 \rightarrow$ abs. min

$f(0,2) = 4$

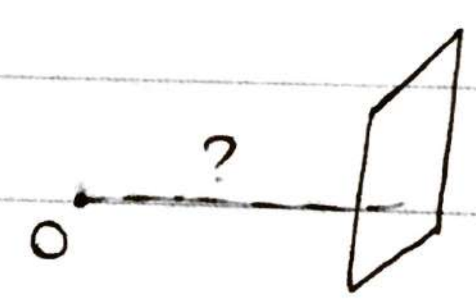
$f(1,2) = 7 \rightarrow$ abs. max

$f(4,4) = -11$

$f(0,9) = -45$

14.8 → LAGRANGE MULTIPLIERS

ex: Find the point on the plane $2x+y-z-5=0$ that is closest to the origin.



$d(x,y,z)$ = distance between (x,y,z) and $(0,0,0)$

$$= \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$$

$$= \sqrt{x^2+y^2+z^2}$$

min $d(x,y,z)$ on $2x+y-z-5=0$

$\downarrow$

$z = 2x+y-5$

First way

Find minimum of $f(x,y) = d(x,y,2x+y-5)$

$$= \sqrt{\underbrace{x^2+y^2+(2x+y-5)^2}_{\star}}$$

$$f_x = \frac{1}{2\sqrt{x}}\left(2x + 2(2x+y-5)\cdot 2\right) = \frac{1}{\sqrt{\star}}(5x+y-5)$$

$$f_y = \frac{1}{2\sqrt{\star}}(2y+2$$